Measure Theory with Ergodic Horizons
Lecture 27

Remark. When μ and ν are σ-finite measures on a measurable space $(X, \mathcal{B})$, and $\mu \ll \nu$, then the Radon-Nikodym derivative $\frac{d\mu}{d\nu}$ is ν-integrable $\iff$ μ is finite; indeed, $\mu(X) = \int_X f \, d\nu$. However, when μ is not finite but σ-finite, we only get that $X = \bigcup_n X_n$ such that $\mu(X_n) < \infty$, so $\int_{X_n} f \, d\mu$ is finite.

This remark motivates the following definition:

Def. Let X be a topological space and let μ be a Borel measure on X. Call a μ-measurable function $f : X \to \overline{\mathbb{R}}$ **locally integrable** if $\mu_{|f|}$ is a locally finite measure. We denote the space of locally integrable functions by $L^1_{loc}(X, \mu)$.

Remark. From a previous proposition about locally finite measures, we get that for all locally compact second ctbl topological spaces, TFAE:

(1) f is locally integrable

(2) $\exists\, X = \bigcup_{n \in \mathbb{N}} U_n$, U_n open, such that $\|f \cdot \mathbb{1}_{U_n}\|_1 < \infty$.

(3) $\|f \cdot \mathbb{1}_K\| < \infty$ for all compact $K \subseteq X$.

In particular, this holds for $\mathbb{R}^d$.

Radon-Nikodym derivatives wrt Lebesgue measure and Lebesgue differentiation.

Let λ denote the Lebesgue measure on $\mathbb{R}^d$ and let μ be another locally finite Borel measure on $\mathbb{R}^d$ such that $\mu \ll \lambda$. Then we know that $\mu = \lambda_f$, where $f = \frac{d\mu}{d\lambda}$, but how to find this f explicitly?

We know that for any $x \in \mathbb{R}^d$, if $B_r(x)$ denotes the d_∞-ball of radius r around x, then $\mu(B_r(x)) = \int_{B_r(x)} f \, d\lambda$ so $\frac{\mu(B_r(x))}{\lambda(B_r(x))} = \frac{1}{\lambda(B_r(x))} \int_{B_r(x)} f \, d\lambda$.
Since $\frac{d\mu}{d\lambda}$ measures the ratio of μ over λ "locally at x", it is conceivable to guess that maybe $\lim_{r \to 0} \frac{\mu(B_r(x))}{\lambda(B_r(x))} = \frac{d\mu}{d\lambda}(x)$. This turns out to be true, i.e. we may "differentiate" μ with respect to λ.

Lebesgue Differentiation Theorem. Let $f \in L^1_{loc}(\mathbb{R}^d, \lambda)$. Then for a.e. $x \in \mathbb{R}^d$,

$$\lim_{r \to 0} \frac{1}{\lambda(B_r(x))} \int_{B_r(x)} f \, d\lambda = f(x).$$

In particular, for any locally finite Borel measure μ on $\mathbb{R}^d$ with $\mu \ll \lambda$,

$$\frac{d\mu}{d\lambda}(x) = \lim_{r \to 0} \frac{\mu(B_r(x))}{\lambda(B_r(x))} \quad \text{a.e.}$$

The "in particular" part is clear from the main part applied to $f := \frac{d\mu}{d\lambda}$, as discussed above, and we focus on proving the main part.
The format of the statement is similar to that of the pointwise ergodic theorem and actually the proof will be similar as well.

Let $A_r f(x) := \frac{1}{\lambda(B_r(x))} \int_{B_r(x)} f \, d\lambda$ denote the averaging operator at radius r.
We want to prove $\lim_{r \to 0} A_r f = f$ a.e.

Lemma. If $g : \mathbb{R}^d \to \mathbb{R}$ is a continuous loc. integrable function then

$$\lim_{r \to 0} A_r g = g \text{ everywhere.}$$

Proof. $|A_r f(x) - f(x)| = \frac{1}{\lambda(B_r(x))} \left| \int (f(y) - f(x)) \, d\lambda \right| \leq \frac{1}{\lambda(B_r(x))} \int_{B_r(x)} |f(y) - f(x)| \, d\lambda$

$$\leq \sup_{y \in B_r(x)} |f(y) - f(x)| \xrightarrow[r \to 0]{} 0 \text{ by continuity.}$$ □

Local-global bridge lemma. Let $f \in L^1(\mathbb{R}^d, \lambda)$. For each $r > 0$, we have:
(a) $\int f \, d\lambda = \int A_r f \, d\lambda$.
(b) $\|A_r f\|_1 \leq \|f\|_1$.
Proof. Part (b) follows from (a) by triangle inequality for integrals, while (a) follows by Fubini and is left as HW. □

Proof of Lebesgue differentiation. WLOG, we may assume $f \in L^1(\mathbb{R}^d, \lambda)$ by replacing f with $\mathbb{1}_{B_n(\vec{0})} \cdot f$; indeed, if the theorem holds for every $n \in \mathbb{N}$ for $\mathbb{1}_{B_n(\vec{0})} \cdot f$, then by combining the null set Z_n for all $n \in \mathbb{N}$, we still get a null set, hence the theorem holds for f a.e. $x \in \mathbb{R}^d$ (also note that if $x \in B_n(\vec{0})$ then $B_r(x) \cap B_{n+1}(\vec{0}) = B_r(x)$ for all $r \leq 1$).

It is enough to show that $D_0 := \{x \in \mathbb{R}^d : |\overbrace{\limsup_{r \to 0} A_r f(x)}^{A^* f(x)} - f(x)| > 0\}$ is null, because then, applying this to $-f$, we will get the same with liminf. Because $D_0 = \bigcup_{n \in \mathbb{N}^+} D_{1/n}(A^* f - f)$, where

$$D_\alpha(h) := \{x \in \mathbb{R}^d : |h| > \alpha\},$$

it is enough to show that $D_\alpha(A^* f - f)$ is null for each $\alpha > 0$. So fix $\alpha > 0$ and $\varepsilon > 0$ in order to show that $\lambda(D_\alpha(A^* f - f)) \leq \varepsilon$.

Recall by a HW exercise that continuous functions are dense in L^1 and let $g : \mathbb{R}^d \to \mathbb{R}$ be a continuous function in L^1 with $\|f - g\|_1 \leq \delta$ for some $\delta > 0$ depending on ε and α, to be chosen later. Note that:

$$|A^* f - f| = |A^* f - A^* g + A^* g - g + g - f| \leq |A^* f - A^* g| + |\underbrace{A^* g - g}_{=0}| + |g - f| \leq |A^*(f - g)| + |g - f|,$$

thus, $D_\alpha(A^* f - f) \subseteq \overbrace{D_{\alpha/2}(A^*(f-g))}^{(b)} \cup \overbrace{D_{\alpha/2}(g - f)}^{(a)}$. Thus, it's enough to show that each of the sets (a) and (b) have measure $\leq \varepsilon/2$.

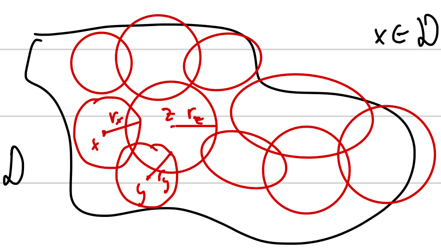
(a) By Chebyshev, $\frac{\alpha}{2} \cdot \lambda(D_{\alpha/2}(g-f)) \leq \|g - f\|_1 \leq \delta$, so $\lambda(D_{\alpha/2}(g-f)) \leq \frac{2\delta}{\alpha}$

and hence if we choose δ so that $2\delta/\alpha \leq \varepsilon/2$, then $\lambda(D_{\alpha/2}(g,f)) \leq \varepsilon/2$.

(b) We would have liked if the same was true for $A^*(f-g)$, i.e. if we had
$$\alpha/2 \cdot \lambda(D_{\alpha/2}(A^*(f-g))) \leq \|A^*(f-g)\|_1$$
because then by the bridge lemma, $\|A^*(f-g)\|_1 \leq \|f-g\|_1 \leq \delta$. Note that if we even prove that
$$\alpha/2 \; \lambda(D_{\alpha/2}(A^*(f-g))) \leq C \cdot \|f-g\|_1 \qquad (*)$$
for some constant C, this would also be enough since then
$$\lambda(D_{\alpha/2}(A^*(f-g))) \leq \alpha/2 \cdot C \cdot \delta,$$
hence choosing δ small enough so that $\alpha/2 \cdot C \cdot \delta \leq \varepsilon/2$, would finish the proof.
Thus, it suffices to prove the following, which gives $(*)$ with $C := 3^d$:

Hardy-Littlewood Maximal Theorem. Let $h \in L^1(\mathbb{R}^d, \lambda)$ and $\alpha > 0$. Then
$$\alpha \cdot \lambda(D_\alpha(A^* h)) \leq 3^d \|h\|_1.$$
In fact, $\alpha \cdot \lambda(D_\alpha(\bar{A} h)) \leq 3^d \|h\|_1$, where $\bar{A}h := \sup_{r \leq 1} A_r|h|$ is the so-called Hardy-Littlewood maximal function.

Proof. Denote $D := D_\alpha(\bar{A}h) := \{x \in \mathbb{R}^d : \sup_{r \leq 1} A_r|h| > \alpha\}$ and note that for each $x \in D$ there is $0 < r_x \leq 1$ such that $A_{r_x}|h| > \alpha$, equivalently:
$$\int_{B_{r_x}(x)} |h| \, d\lambda \geq \alpha \cdot \lambda(B_{r_x}(x)).$$



We want to deduce from this that $\alpha \cdot \lambda(D) \leq \int_D |h| \, d\lambda$, up to a constant multiple...